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The Künneth formula in Floer homology for manifolds with restricted contact type boundary

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Abstract. We prove the Künneth formula in Floer (co)homology for manifolds with restricted contact type boundary. We use Viterbo's definition of Floer homology, involving the symplectic completion by adding a positive cone over the boundary. The Künneth formula implies the vanishing of Floer (co)homology for subcritical Stein manifolds. Other applications include the Weinstein conjecture in certain product manifolds, obstructions to exact Lagrangian embeddings, existence of holomorphic curves with Lagrangian boundary condition, as well as symplectic capacities.

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1. Introduction

The present paper is concerned with the Floer homology groups $FH_*(M)$ of a compact symplectic manifold (M, ω) with contact type boundary, as well as with their cohomological dual analogues $FH^*(M)$. The latter were defined by Viterbo in [V] and are invariants that take into account the topology of the underlying manifold *and*, through an algebraic limit process, all closed characteristics on ∂M. Their definition is closely related to the Symplectic homology groups of Floer, Hofer, Cieliebak and Wysocki [FH, CFH, FHW, CFHW, C1].

Throughout this paper we will assume that ω is exact, and in particular $\langle \omega, \pi_2(M)\rangle = 0$. This last condition will be referred to as *symplectic asphericity*. The groups $FH_*(M)$ are invariant with respect to deformations of the symplectic form ω that preserve the contact type character of the boundary and the condition $\langle \omega, \pi_2(M)\rangle = 0$. The groups $FH_*(M)$ actually depend only on the *symplectic completion* $\widehat{M}$ of M. The manifold $\widehat{M}$ is obtained by gluing a positive cone along the boundary ∂M and carries a symplectic form $\widehat{\omega}$ which is canonically determined by ω and the conformal vector field on M. We shall often write $FH_*(\widehat{M})$ instead of $FH_*(M)$. The grading on $FH_*(\widehat{M})$ is given by minus the Conley-Zehnder index

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modulo 2ν, with ν the minimal Chern number of M. There exist canonical maps

$$H_{n+*}(M,\ \partial M) \xrightarrow{c_*} FH_*(\widehat{M}),$$

$$FH^*(\widehat{M}) \xrightarrow{c^*} H^{n+*}(M,\ \partial M)$$

which shift the grading by $n = \frac{1}{2}\dim M$.

Theorem A (Künneth formula). *Let $(M^{2m},\ \omega)$ and $(N^{2n},\ \sigma)$ be compact symplectic manifolds with* restricted *contact type boundary. Denote the minimal Chern numbers of M, N and $M \times N$ by ν_M, ν_N and $\nu_{M\times N} = \gcd(\nu_M,\ \nu_N)$ respectively.*

a. For any ring A of coefficients there exists a short exact sequence which splits noncanonically

$$\begin{tikzcd} \bigoplus_{\widehat{r}+\widehat{s}=k} FH_{\widehat{r}}(M,\ \omega)\otimes FH_{\widehat{s}}(N,\ \sigma) \arrow[r, tail] & FH_k(M\times N,\ \omega\oplus\sigma) \arrow[d, two heads] \\ & \bigoplus_{\widehat{r}+\widehat{s}=k-1} \mathrm{Tor}_1^A\big(FH_{\widehat{r}}(M,\ \omega),\ FH_{\widehat{s}}(N,\ \sigma)\big) \end{tikzcd} \quad (1)$$

The morphism c_* induces a morphism of exact sequences whose source is the Künneth exact sequence of the product $(M,\ \partial M)\times(N,\ \partial N)$ and whose target is (1).

b. For any field $\mathbb{K}$ of coefficients there is an isomorphism

$$\bigoplus_{\widehat{r}+\widehat{s}=k} FH^{\widehat{r}}(M,\ \omega)\otimes FH^{\widehat{s}}(N,\ \sigma) \xrightarrow{\ \sim\ } FH^k(M\times N,\ \omega\oplus\sigma)\,, \quad (2)$$

The morphism c^* induces a commutative diagram with respect to the Künneth isomorphism in cohomology for $(M,\ \partial M)\times(N,\ \partial N)$.

In the above notation we have $k \in \mathbb{Z}/2\nu_{M\times N}\mathbb{Z}$, $0 \le r \le 2\nu_M - 1$, $0 \le s \le 2\nu_N - 1$ and the $\widehat{\ }$ symbol associates to an integer its class in the corresponding $\mathbb{Z}/2\nu\mathbb{Z}$ ring. The reader can consult [D, VI.12.16] for a construction of the Künneth exact sequence in singular homology.

The algebraic properties of the map c^* strongly influence the symplectic topology of the underlying manifold. Our applications are based on the following theorem, which summarizes part of the results in [V].

Theorem (Viterbo [V]).

Let $(M^{2m},\ \omega)$ be a manifold with contact type boundary such that $\langle\omega,\ \pi_2(M)\rangle = 0$. Assume the map $c^* : FH^*(M) \longrightarrow H^{2m}(M,\ \partial M)$ is not *surjective. Then the following hold.*

a. The same is true for any hypersurface of restricted contact type $\Sigma \subset M$ bounding a compact region;

b. Any hypersurface of contact type $\Sigma \subset M$ bounding a compact region carries a closed characteristic (Weinstein conjecture);

c. There is no exact Lagrangian embedding $L \subset M$ (here M is assumed to be exact by definition);

d. *For any Lagrangian embedding $L \subset M$ there is a loop on L which is contractible in M, has strictly positive area and whose Maslov number is at most equal to $m+1$;*
e. *For any Lagrangian embedding $L \subset M$ and any compatible almost complex structure J there is a nonconstant J-holomorphic curve S (of unknown genus) with non-empty boundary $\partial S \subset L$.*

Viterbo [V] introduces the following definition, whose interest is obvious in the light of the above theorem.

Definition 1. (Viterbo) *A symplectic manifold (M^{2m}, ω) which verifies $\langle \omega, \pi_2(M) \rangle = 0$ is said to satisfy the Strong Algebraic Weinstein Conjecture (SAWC) if the composed morphism below is not surjective*

$$FH^*(M) \xrightarrow{c^*} H^*(M, \partial M) \xrightarrow{\mathrm{pr}} H^{2m}(M, \partial M) .$$

We shall still denote the composed morphism by c^*. Theorem A now implies that the property of satisfying the SAWC is stable under products, with all the geometric consequences listed above.

Theorem B. *Let (M, ω) be a symplectic manifold with restricted contact type boundary satisfying the SAWC. Let (N, σ) be an arbitrary symplectic manifold with restricted contact type boundary. The product $(\widehat{M} \times \widehat{N}, \widehat{\omega} \oplus \widehat{\sigma})$ satisfies the SAWC and assertions (a) to (e) in the above theorem of Viterbo hold. In particular, the Weinstein conjecture holds and there is no exact Lagrangian embedding in $\widehat{M} \times \widehat{N}$.*

The previous result can be applied for subcritical Stein manifolds of finite type. These are complex manifolds $\widehat{M}$ which admit proper and bounded from below plurisubharmonic Morse functions with only a finite number of critical points, all of index strictly less than $\frac{1}{2} \dim_{\mathbb{R}} \widehat{M}$ [El]. They satisfy the SAWC as proved by Viterbo [V]. Cieliebak [C1] has proved that their Floer homology actually vanishes. We can recover this through Theorem A by using another of his results [C2], namely that every such manifold is Stein deformation equivalent to a split one $(V \times \mathbb{C}, \omega \oplus \omega_{\mathrm{std}})$. This can be seen as an extension of the classical vanishing result $FH_*(\mathbb{C}^\ell) = 0$, $\ell \geq 1$ [FHW].

Theorem C (Cieliebak [C1]). *Let $\widehat{M}$ be a subcritical Stein manifold of finite type. Its Floer homology vanishes*

$$FH_*(\widehat{M}) = 0 .$$

The paper is organized as follows. In Section 2 we state the relevant definitions and explain the main properties of the invariant FH_*.Section 3 contains the proof of Theorem A. The proofs of Theorems B and C, together with other applications, are gathered in Section 4.

Let us point out where the difficulty lies in the proof of Theorem A. Floer homology is defined on closed manifolds for any Hamiltonian satisfying some generic nondegeneracy condition, and this condition is *stable* under sums $H(x)+K(y)$ on products $M\times N \ni (x, y)$. This trivially implies (with field coefficients) a Künneth formula of the type $FH_*(M\times N;\ H+K,\ J_1\oplus J_2) \simeq FH_*(M;\ H,\ J_1)\otimes FH_*(N;\ K,\ J_2)$. On the other hand, Floer homology for manifolds with contact type boundary is defined using Hamiltonians with a rigid behaviour at infinity and involves an algebraic limit construction. This class of Hamiltonians is *not stable* under the sum operation $H(x) + K(y)$ on $M \times N$. One may still define Floer homology groups $FH_*(M \times N;\ H + K,\ J_1 \oplus J_2)$, but the resulting homology might well be different, in the limit, from $FH_*(\widehat{M} \times \widehat{N})$. The whole point of the proof is to show that this is not the case.

This paper is the first of a series studying the Floer homology of symplectic fibrations with contact type boundary. It treats trivial fibrations with open fiber and base. A spectral sequence of Leray-Serre type for symplectic fibrations with closed base and open fiber is constructed in [O3].

2. Definition of Floer homology

Floer homology has been first defined by A. Floer for closed manifolds in a series of papers [F1,F2] which proved Arnold's conjecture for a large class of symplectic manifolds including *symplectically aspherical* ones. In this situation Floer's construction can be summarized as follows. Consider a periodic time-dependent Hamiltonian $H : \mathbb{S}^1 \times M \longrightarrow \mathbb{R}$ with Hamiltonian vector field X_H^t defined by $\iota_{X_H^t}\omega = dH(t, \cdot)$. The associated *action functional*

$$A_H : C^\infty_{\mathrm{contr}}(M) \longrightarrow \mathbb{R}\ ,$$

$$\gamma \longmapsto -\int_{D^2} \bar{\gamma}^*\omega - \int_{\mathbb{S}^1} H(t,\ \gamma(t))dt$$

is defined on the space of smooth contractible loops

$$C^\infty_{\mathrm{contr}}(M) = \left\{\gamma : \mathbb{S}^1 \longrightarrow M\ :\ \exists \text{ smooth } \bar{\gamma} : D^2 \longrightarrow M,\ \bar{\gamma}|_{\mathbb{S}^1} = \gamma\right\}.$$

The critical points of A_H are precisely the 1-periodic solutions of $\dot{\gamma} = X_H^t(\gamma(t))$, and we denote the corresponding set by $\mathcal{P}(H)$. We suppose that the elements of $\mathcal{P}(H)$ are nondegenerate i.e. the time one return map has no eigenvalue equal to 1. Each such periodic orbit γ has a $\mathbb{Z}/2\nu\mathbb{Z}$-valued Conley-Zehnder index $i_{CZ}(\gamma)$ (see [RS] for the definition), where ν is the *minimal Chern number*. The latter is defined by $\langle c_1,\ \pi_2(M)\rangle = \nu\mathbb{Z}$ and by the convention $\nu = \infty$ if $\langle c_1,\ \pi_2(M)\rangle = 0$.

Let also J be a compatible almost complex structure. The homological Floer complex is defined as

$$FC_k(H, J) = \bigoplus_{\substack{\gamma \in \mathcal{P}(H) \\ i_{CZ}(\gamma) = -k \mod 2\nu}} \mathbb{Z}\langle\gamma\rangle ,$$

$$\delta : FC_k(H, J) \longrightarrow FC_{k-1}(H, J) ,$$

$$\delta\langle\gamma\rangle = \sum_{\substack{\gamma' \in \mathcal{P}(H) \\ i_{CZ}(\gamma') = -k+1 \mod 2\nu}} \#\mathcal{M}(\gamma, \gamma'; H, J)/\mathbb{R} . \tag{3}$$

Here $\mathcal{M}(\gamma, \gamma'; H, J)$ denotes the space of trajectories for the negative L^2 gradient of A_H with respect to the metric $\omega(\cdot, J\cdot)$, running from γ to γ':

$$\mathcal{M}(\gamma, \gamma'; H, J) = \left\{ u : \mathbb{R} \times \mathbb{S}^1 \longrightarrow M \ : \ \begin{array}{c} u_s + J \circ u \cdot u_t - \nabla H(t, u) = 0 \\ u(s, \cdot) \longrightarrow \gamma, \ s \longrightarrow -\infty \\ u(s, \cdot) \longrightarrow \gamma', \ s \longrightarrow +\infty \end{array} \right\}. \tag{4}$$

The additive group $\mathbb{R}$ acts on $\mathcal{M}(\gamma, \gamma'; H, J)$ by translations in the s variable, while the symbol $\#\mathcal{M}(\gamma, \gamma'; H, J)/\mathbb{R}$ stands for an algebraic count of the elements of $\mathcal{M}(\gamma, \gamma'; H, J)/\mathbb{R}$. We note that it is possible to choose any coefficient ring instead of $\mathbb{Z}$ once the sign assignment procedure is available.

The crucial statements of the theory are listed below. The main point is that the symplectic asphericity condition prevents the loss of compactness for Floer trajectories by bubbling-off of nonconstant J-holomorphic spheres: the latter simply cannot exist.

a. under the nondegeneracy assumption on the 1-periodic orbits of H and for a generic choice of J, the space $\mathcal{M}(\gamma, \gamma'; H, J)$ is a manifold of dimension $i_{CZ}(\gamma') - i_{CZ}(\gamma) = -i_{CZ}(\gamma) - (-i_{CZ}(\gamma'))$. If this difference is equal to 1, the space $\mathcal{M}(\gamma, \gamma'; H, J)/\mathbb{R}$ consists of a finite number of points. Moreover, there is a consistent choice of signs for these points with respect to which $\delta \circ \delta = 0$;
b. the Floer homology groups $FH_*(H, J)$ are independent of H and J. More precisely, for any two pairs (H_0, J_0) and (H_1, J_1) there is a generic choice of a smooth homotopy (H_s, J_s), $s \in \mathbb{R}$ with $(H_s, J_s) \equiv (H_0, J_0)$, $s \leq 0$, $(H_s, J_s) \equiv (H_1, J_1)$, $s \geq 1$ defining a map

$$\sigma^{(H_0, J_0)}_{(H_1, J_1)} : FC_k(H_0, J_0) \longrightarrow FC_k(H_1, J_1) , \tag{5}$$

$$\sigma\langle\gamma\rangle = \sum_{\substack{\gamma' \in \mathcal{P}(H_1) \\ i_{CZ}(\gamma') = i_{CZ}(\gamma) = -k \mod 2\nu}} \#\mathcal{M}(\gamma,\, \gamma';\, H_s,\, J_s)\ .$$

The notation $\mathcal{M}(\gamma,\, \gamma';\, H_s,\, J_s)$ stands for the space of solutions of the equation

$$u_s + J(s,\, u(s,\, t))u_t - \nabla H(s,\, t,\, u(s,\, t)) = 0\ , \tag{6}$$

which run from γ to γ'. The map $\sigma^{(H_0,\, J_0)}_{(H_1,\, J_1)}$ induces an isomorphism in cohomology and this isomorphism is independent of the choice of the homotopy. We shall call it in the sequel the "continuation morphism".

c. if H is time independent, Morse and sufficiently small in some C^2 norm, then the 1-periodic orbits of X_H are the critical points of H and the Morse and Conley-Zehnder indices satisfy $i_{\mathrm{Morse}}(\gamma) = m + (-i^0_{CZ}(\gamma))$, $m = \frac{1}{2}\dim M$. Here $i^0_{CZ}(\gamma)$ is the Conley-Zehnder computed with respect to the trivial filling disc. Moreover, the Floer trajectories running between points with index difference equal to 1 are independent of the t variable and the Floer complex is equal, modulo a shift in the grading, with the Morse complex of H.
We infer that for any regular pair $(H,\, J)$ we have

$$FH_k(H,\, J) \simeq \bigoplus_{l \equiv k \mod 2\nu} H_{l+m}(M)\ ,\ k \in \mathbb{Z}/2\nu\mathbb{Z}\ .$$

Remark. An analogous construction yields a *cohomological complex* by considering in (3) the space of trajectories $\mathcal{M}(\gamma',\, \gamma;\, H,\, J)$.

We explain now how the above ideas can be adapted in order to construct a symplectic invariant for manifolds with contact type boundary. We follow [V], but similar constructions can be found in [FH, CFH, C1] (see [O2] for a survey).

Definition 2. *A symplectic manifold $(M,\, \omega)$ is said to have a* contact type boundary *if there is a vector field X defined in a neighbourhood of ∂M, pointing outwards and transverse to ∂M, which satisfies*

$$L_X\omega = \omega\ .$$

We say that M has restricted contact type boundary *if X is defined globally.*

We call X and $\lambda = \iota(X)\omega$ the *Liouville vector field* and the *Liouville form* respectively, with $d\lambda = \omega$. We define the Reeb vector field X_{Reeb} as the generator of $\ker \omega|_{T\partial M}$ normalized by $\lambda(X_{\mathrm{Reeb}}) = 1$. An integral curve of X_{Reeb} is called a *characteristic*. We have $\varphi_t^*\omega = e^t\omega$ and this implies that a neighbourhood of ∂M is foliated by the hypersurfaces $\big(\varphi_t(\partial M)\big)_{-\epsilon \le t \le 0}$, whose characteristic dynamics are conjugate.

The Floer homology groups $FH_*(M)$ of a manifold with contact type boundary are an invariant that takes into account:

- the dynamics on the boundary by "counting" characteristics of arbitrary period;
- the interior topology of M by "counting" interior 1-periodic orbits of Hamiltonians.

In order to understand its definition, let us explain how one can "see" characteristics of arbitrary period by using 1-periodic orbits of special Hamiltonians. There is a symplectic diffeomorphism onto a neighbourhood $\mathcal{V}$ of the boundary

$$\Psi : \big(\partial M \times [1-\delta,\ 1],\ d(S\lambda|)\big) \longrightarrow (\mathcal{V},\ \omega), \quad \delta > 0 \text{ small},$$

$$\Psi(p,\ S) = \varphi_{\ln(S)}(p)\ ,$$

where $\lambda|$ denotes the restriction of λ to ∂M (we actually have $\Psi^*\lambda = S\lambda|$). We define the *symplectic completion*

$$(\widehat{M},\ \widehat{\omega}) = (M,\ \omega) \cup_\Psi \big(\partial M \times [1,\ \infty[,\ d(S\lambda|)\big)\ .$$

Consider now Hamiltonians H such that $H(p,\ S) = h(S)$ for $S \geq 1-\delta$, where $h : [1-\delta,\ 1] \longrightarrow \mathbb{R}$ is smooth. It is straightforward to see that

$$X_H(p,\ S) = -h'(S)X_{\mathrm{Reeb}}, \quad S \geq 1-\delta\ .$$

The 1-periodic orbits of X_H that are located on the level S correspond to characteristics on ∂M having period $h'(S)$ under the parameterization given by $-X_{\mathrm{Reeb}}$. The general principle that can be extracted out of this computation is that *one "sees" more and more characteristics as the variation of h is bigger and bigger*.

We define a Hamiltonian to be *admissible* if it satisfies $H \leq 0$ on M and it is of the form $H(p,\ S) = h(S)$ for S big enough[1], with h convex increasing and such that there exists $S_0 \geq 1$ with h' constant for $S \geq S_0$. We call such a Hamiltonian *linear at infinity*. Moreover, we assume that the slope at infinity of h is not the area of a closed characteristic on ∂M and that all 1-periodic orbits of H are nondegenerate. One method to obtain such Hamiltonians is to slightly perturb functions $h(S)$, where $h : [1-\delta,\ \infty[\longrightarrow \mathbb{R}$ is equal to zero in a neighbourhood of $1-\delta$, strictly convex on $\{h > 0\}$ and linear at infinity with slope different from the area of any closed characteristic, by a perturbation localized around the periodic orbits. We point out that there are admissible Hamiltonians having arbitrarily large values of the slope at infinity.

The *admissible almost complex structures* are defined to be those which satisfy the following conditions for *large enough values of S*:

$$\begin{cases} J_{(p,\,S)}|_\xi = J_0, \\ J_{(p,\,S)}\big(\frac{\partial}{\partial S}\big) = \frac{1}{CS}X_{\mathrm{Reeb}}(p), \quad C > 0, \\ J_{(p,\,S)}(X_{\mathrm{Reeb}}(p)) = -CS\frac{\partial}{\partial S}\ , \end{cases}$$

[1] Our definition is slightly more general than the one in [V] in that we prescribe the behaviour of admissible Hamiltonians only near infinity and not on the whole of $S \geq 1$. It is clear that the a priori C^0 bounds require no additional argument than the one in [V]. The situation is of course different if one enlarges further the admissible class.

where J_0 is an almost complex structure compatible with the restriction of ω to the contact distribution $\xi = \ker \lambda|$ on ∂M. These are precisely the almost complex structures which are invariant under homotheties $(p, S) \longmapsto (p, aS)$, $a > 0$ for large enough values of S.

The crucial fact is that the function $(p, S) \longmapsto S$ is plurisubharmonic with respect to this class of almost complex structures. This means that $d(dS \circ J)(v, Jv) < 0$ for any nonzero $v \in T_{(p, S)}\big(\partial M \times [1, \infty[\big)$, $p \in \partial M$, $S \geq 1$ big enough, and indeed we have $d(dS \circ J) = d(-CS\lambda|) = -C\widehat{\omega}$. Plurisubharmonicity implies that for any J-holomorphic curve $u : D^2 \longrightarrow \partial M \times [S_0, \infty[$, $S_0 \geq 1$ big enough one has $\Delta(S \circ u) \geq 0$. In particular the maximum of u is achieved on the boundary ∂D^2 (see for example [GiTr], Theorem 3.1). A similar argument applies to solutions of the Floer equation (4), as well as to solutions of the parameterized Floer equation (6) for *increasing homotopies* satisfying $\frac{\partial^2 h}{\partial s \partial S} \geq 0$ [O2]. This implies that solutions are contained in an a priori determined compact set. All compactness results in Floer's theory therefore carry over to this situation and so does the construction outlined for closed manifolds.

We introduce a partial order on regular pairs (H, J) as

$$(H, J) \prec (K, \widetilde{J}) \qquad \text{iff} \qquad H \leq K \ .$$

The continuation morphisms (5) form a direct system with respect to this order and we define the Floer homology groups as

$$FH_*(M) = \lim_{\substack{\rightarrow \\ (H, J)}} FH_*(H, J) \ .$$

An important refinement of the definition consists in using a truncation by the values of the action. The latter is decreasing along Floer trajectories and one builds a 1-parameter family of subcomplexes of $FC^*(H, J)$, defined as

$$FC_k^{]-\infty, a[}(H, J) = \bigoplus_{\substack{\gamma \in \mathcal{P}(H) \\ i_{CZ}(\gamma) = -k \mod 2\nu \\ A_H(\gamma) < a}} \mathbb{Z}\langle \gamma \rangle \ .$$

This allows one to define the corresponding quotient complexes

$$FC_*^{[a, b[}(H, J) = FC_*^{]-\infty, b[}(H, J) / FC_*^{]-\infty, a[}(H, J), \quad -\infty \leq a < b \leq \infty$$

and the same direct limit process goes through. We therefore put

$$FH_*^{[a, b[}(M) = \lim_{\substack{\rightarrow \\ (H, J)}} FH_*^{[a, b[}(H, J) \ .$$

Let us now make a few remarks on the properties of the above invariants.

a. there is a natural cofinal family of Hamiltonians whose values of the action on the 1-periodic orbits is positive or arbitrarily close to 0 (see the construction of H_1 in Figure 1 (1) described in Section 3). This implies that $FH_*^{[a,\,b[}(M) = 0$ if $b < 0$ and $FH_*^{[a,\,b[}$ does not depend on a if the latter is strictly negative. In particular we have

$$FH_*(M) = FH_*^{[a,\,\infty[}(M), \qquad a < 0\,.$$

b. the infimum T_0 of the areas of closed characteristics on the boundary is always strictly positive and therefore

$$FH_k^{[a,\,\epsilon[}(M) \simeq \bigoplus_{l \equiv k \mod 2\nu} H_{l+m}(M,\,\partial M),$$

where $a < 0 \le \epsilon < T_0, k \in \mathbb{Z}/2\nu\mathbb{Z}, m = \frac{1}{2}\dim M$. This follows from the fact that, in the limit, the Hamiltonians become C^2-small on $M \setminus \partial M$ and the Floer complex reduces to a Morse complex that computes the relative cohomology, as $-\nabla H$ points inward along ∂M.

c. there are obvious truncation morphisms

$$FH_*^{[a,\,b[}(H,\,J) \longrightarrow FH_*^{[a',\,b'[}(H,\,J),\ a \le a',\ b \le b'$$

which induce morphisms $FH_*^{[a,\,b[}(M) \longrightarrow FH_*^{[a',\,b'[}(M)$, $a \le a'$, $b \le b'$. If $a = a' < 0$, $0 \le b < T_0$ and $b' = \infty$ we obtain a natural morphism

$$\bigoplus_{l \equiv k \mod 2\nu} H_{l+m}(M,\,\partial M) \xrightarrow{c_*} FH_k(M)\,,$$

or, written differently,

$$H_*(M,\,\partial M) \xrightarrow{c_*} FH_*(M)\,. \tag{7}$$

We also note at this point that we have

$$FH_*(M) = \lim_{\substack{\rightarrow \\ b}} \lim_{\substack{\rightarrow \\ (H,\,J)}} FH_*^{[a,\,b[}(H,\,J), \qquad a < 0$$

and that the two limits above can be interchanged by general properties of bi-directed systems.

Fundamental principle (Viterbo [V]). *If the morphism $H_*(M, \partial M) \xrightarrow{c_*} FH_*(M)$ is not bijective, then there is a closed characteristic on ∂M.* Indeed, either there is some extra generator in $FH_*(M)$, or some Morse homological generator of $H_*(M, \partial M)$ is killed in $FH_*(M)$. The "undesired guest" in the first case or the "killer" in the second case necessarily corresponds to a closed characteristic on ∂M.

The version of Floer homology that we defined above has various invariance properties [V]. The main one that we shall use is the following.

Proposition 1. *The Floer homology groups $FH_*(M)$ are an invariant of the completion $\widehat{M}$ in the following sense: for any open set with smooth boundary $U \subset M$ such that $\partial U \subset \partial M \times [1, \infty[$ and the Liouville vector field $S\frac{\partial}{\partial S}$ is transverse and outward pointing along ∂U, we have*

$$FH_*(\widehat{M}) \simeq FH_*(\widehat{U})\ .$$

Proof. One can realise a differentiable isotopy between M and U along the Liouville vector field. This corresponds to an isotopy of symplectic forms on M starting from the initial form $\omega = \omega_0$ and ending with the one induced from U, denoted by ω_1. During the isotopy the boundary ∂M remains of contact type and the symplectic asphericity condition is preserved. An invariance theorem of Viterbo [V] shows that $FH_*(M, \omega) \simeq FH_*(M, \omega_1)$. On the other hand $FH_*(M, \omega_1) \simeq FH_*(U, \omega)$ because (M, ω_1) and (U, ω) are symplectomorphic. □

3. Proof of Theorem A

Before beginning the proof, let us note that the natural class of manifolds for which one can *define* Floer homology groups for a product is that of manifolds with restricted contact type boundary. The reason is that $\partial\big(M \times N\big)$ involves the full manifolds M and N, not only some neighbourhoods of their boundaries. If X and Y are the conformal vector fields on M and N respectively and $\pi_M : M \times N \longrightarrow M$, $\pi_N : M \times N \longrightarrow N$ are the canonical projections, the natural conformal vector field on $M \times N$ is $Z = \pi_M^* X + \pi_N^* Y$. In order for Z to be defined in a neighbourhood of $\partial(M \times N)$ it is necessary that X and Y be globally defined.

We only prove a) because the proof of b) is entirely dual. One has to reverse arrows and replace direct limits with inverse limits. The difference in the statement is due to the fact that the inverse limit functor is in general not exact, except when each term of the directed system is a finite dimensional vector space [ES]. For the sake of clarity we shall give the proof under the assumption $\langle c_1(TM), \pi_2(M)\rangle = 0$, $\langle c_1(TN), \pi_2(N)\rangle = 0$, so that the grading on Floer homology is defined over $\mathbb{Z}$.

I. We establish the short exact sequence (1). Here is the sketch of the proof. We consider on $\widehat{M} \times \widehat{N}$ a Hamiltonian of the form $H(t, x) + K(t, y)$, $t \in \mathbb{S}^1$, $x \in \widehat{M}$,

$y \in \widehat{N}$. For an almost complex structure on $\widehat{M} \times \widehat{N}$ of the form $J^1 \oplus J^2$, with J^1, J^2 (generic) almost complex structures on $\widehat{M}$ and $\widehat{N}$ respectively, the Floer complex for $H + K$ can be identified, modulo truncation by the action issues, with the tensor product of the Floer complexes of $(H,\ J^1)$ and $(K,\ J^2)$. Nevertheless, the Hamiltonian $H + K$ is not linear at infinity and hence not admissible. We refer to [O1] for a discussion of the weaker notion of asymptotic linearity and a proof of the fact that $H + K$ does not even belong to this extended admissible class. The main idea of our proof is to construct an admissible pair $(L,\ J)$ whose Floer complex is roughly the same as the one of $(H + K,\ J^1 \oplus J^2)$. The Hamiltonian L will have lots of additional 1-periodic orbits compared to $H + K$, but all these will have negative enough action for them not to be counted in the relevant truncated Floer complexes.

We define the *period spectrum* $\mathcal{S}(\Sigma)$ of a contact type hypersurface Σ in a symplectic manifold as being the set of periods of closed characteristics on Σ, the latter being parameterized by the Reeb flow. We assume from now on that the period spectra of ∂M and ∂N are *discrete and injective* i.e. the periods of the closed characteristics form a strictly increasing sequence, every period being associated to a unique characteristic which is transversally nondegenerate. This property is C^∞-generic among hypersurfaces [T], while Floer homology does not change under a small C^∞-perturbation of the boundary (Proposition 1). Assuming a discrete and injective period spectrum amounts therefore to no loss of generality.

We shall construct cofinal familes of Hamiltonians and almost complex structures $(H_\nu,\ J_\nu^1)$, $(K_\nu,\ J_\nu^2)$, $(L_\nu,\ J_\nu)$ on $\widehat{M}$, $\widehat{N}$ and $\widehat{M} \times \widehat{N}$ respectively, with the following property.

Main Property. *Let $\delta > 0$ be fixed. For any $b > 0$, there is a positive integer $\nu(b,\ \delta)$ such that, for all $\nu \geq \nu(b,\ \delta)$, the following inclusion of differential complexes holds:*

$$\begin{tikzcd}
\bigoplus_{r+s=k} FC_r^{[-\delta,\ \frac{b}{2}[}(H_\nu,\ J_\nu^1) \otimes FC_s^{[-\delta,\ \frac{b}{2}[}(K_\nu,\ J_\nu^2) \arrow[r] & FC_k^{[-\delta,\ b[}(L_\nu,\ J_\nu) \arrow[dl] \\
\bigoplus_{r+s=k} FC_r^{[-\delta,\ 2b[}(H_\nu,\ J_\nu^1) \otimes FC_s^{[-\delta,\ 2b[}(K_\nu,\ J_\nu^2) \arrow[r] & FC_k^{[-\delta,\ 4b[}(L_\nu,\ J_\nu)
\end{tikzcd} \tag{8}$$

It is important to note that we require the two composed arrows

$$FC_k^{[-\delta,\ b[}(L_\nu,\ J_\nu) \hookrightarrow FC_k^{[-\delta,\ 4b[}(L_\nu,\ J_\nu),$$

$$\bigoplus_{r+s=k} FC_r^{[-\delta,\ \frac{b}{2}[}(H_\nu,\ J_\nu^1) \otimes FC_s^{[-\delta,\ \frac{b}{2}[}(K_\nu,\ J_\nu^2) \hookrightarrow \bigoplus_{r+s=k} FC_r^{[-\delta,\ 2b[}(H_\nu,\ J_\nu^1) \otimes FC_s^{[-\delta,\ 2b[}(K_\nu,\ J_\nu^2)$$

to be the usual inclusions corresponding to the truncation by the action. In practice we shall construct *autonomous* Hamiltonians having transversally nondegenerate

1-periodic orbits, but one should think in fact of small local perturbations of these ones, along the technique of [CFHW]. The latter consists in perturbing an autonomous Hamiltonian in the neighbourhood of a transversally nondegenerate 1-periodic orbit γ, replacing γ by precisely two nondegenerate 1-periodic orbits corresponding to the two critical points of a Morse function on the embedded circle given by $\mathrm{im}(\gamma)$. The Conley-Zehnder indices of the perturbed orbits differ by one. Moreover, the perturbation can be chosen arbitrarily small in any C^k-norm and the actions of the perturbed orbits can be brought arbitrarily close to the action of γ.

a. Let S', S'' be the vertical coordinates on $\widehat{M}$ and $\widehat{N}$ respectively. Let (H_ν), (K_ν) be *cofinal* families of autonomous Hamiltonians on $\widehat{M}$ and $\widehat{N}$, such that $H_\nu(p', S') = h_\nu(S')$ for $S' \geq 1$, $K_\nu(p'', S'') = k_\nu(S'')$ for $S'' \geq 1$, with h_ν, k_ν convex and linear of slope λ_ν outside a small neighbourhood of 1. We assumed that the period spectra of ∂M and ∂N are discrete and injective and so we can choose $\lambda_\nu \notin \mathcal{S}(\partial M) \cup \mathcal{S}(\partial N)$ with $\lambda_\nu \longrightarrow \infty$, $\nu \longrightarrow \infty$. We shall drop the subscript ν in the sequel by referring to H_ν, K_ν and λ_ν as H, K and λ. Let us denote

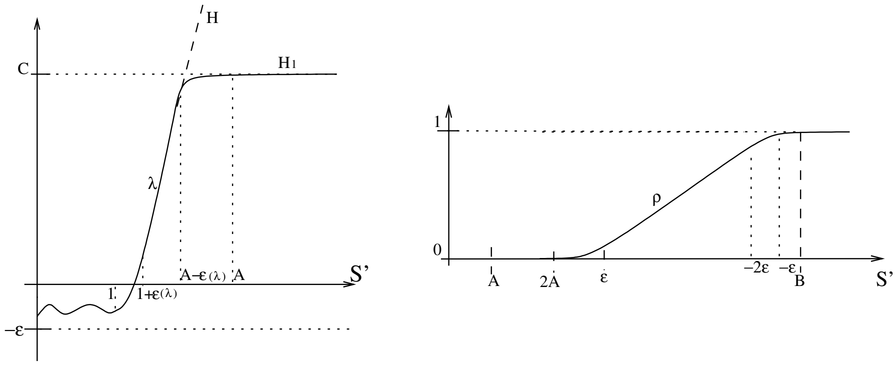
$$\begin{aligned} \eta_\lambda &= \mathrm{dist}\big(\lambda, \mathcal{S}(\partial M) \cup \mathcal{S}(\partial N)\big) > 0 , \\ T_0(\partial M) &= \min \mathcal{S}(\partial M) , \quad T_0(\partial N) = \min \mathcal{S}(\partial N) , \\ T_0 &= \min \big(T_0(\partial M), T_0(\partial N)\big) > 0. \end{aligned}$$

b. Our starting point is the construction by Hermann [He] of a cofinal family which allows one to identify, in the case of bounded open sets with restricted contact type boundary in $\mathbb{C}^n$, the Floer homologies defined by Viterbo [V] and Floer and Hofer [FH]. We fix

$$A = A(\lambda) = 5\lambda/\eta_\lambda > 1$$

and consider the Hamiltonian H_1 equal to H for $S' \leq A - \epsilon(\lambda)$ and constant equal to C for $S' \geq A$, with C arbitrarily close to $\lambda(A-1)$. Here $\epsilon(\lambda)$ is chosen to be small enough and positive. We perform the same construction in order to get a Hamiltonian K_1. We suppose (Figure 1 (1)) that H_1 takes its values in the interval $[-\epsilon, 0)$ on the interior of M where it is also C^2-small and that $H_1(\underline{x}, S') = h(S')$ on $\partial M \times [1, \infty[$, with $h' \equiv \lambda$ on $[1+\epsilon(\lambda), A-\epsilon(\lambda)]$ and $h' \equiv 0$ on $[A, \infty[$, where $\epsilon(\lambda) = \epsilon/\lambda$. Thus H_1 takes values in $[-\epsilon, \epsilon]$ for $S' \in [1, 1+\epsilon(\lambda)]$ and in $[\lambda(A-1) - 2\epsilon, \lambda(A-1)]$ for $S' \in [A - \epsilon(\lambda), A]$.

The Hamiltonian H_1 has additional 1-periodic orbits compared to H. These are either constants on levels $H_1 = C$ with action $-C \simeq -\lambda(A-1)$, or orbits corresponding to characteristics on the boundary, appearing on levels $S = \mathrm{ct.}$ close to A. The action of the latter is arbitrarily close to $h'(S)S - h(S) \leq (\lambda - \eta_\lambda) \cdot A - \lambda(A-1) + 2\epsilon \leq -3\lambda \longrightarrow -\infty$, $\lambda \longrightarrow \infty$. The special choice of A is motivated by the previous computation. We see in particular that it is crucial to



(1) (2)

Fig. 1. The Hamiltonian H_1 and the truncation function ρ

take into account the gap η_λ in order to be able to make the action of this kind of orbits tend to $-\infty$.

c. We deform now $H_1 + K_1$ to a Hamiltonian that is constant equal to $2C$ outside the compact set $\{S' \le B,\ S'' \le B\}$, with

$$B = A\sqrt{\lambda}\ .$$

This already holds in $\{S' \ge A,\ S'' \ge A\}$. We describe the corresponding deformation in $\{S' \le A,\ S'' \ge A\}$ and perform the symmetric construction in $\{S' \ge A,\ S'' \le A\}$. Let us define

$$H_2 : \widehat{M} \times \partial N \times [A,\ \infty[\longrightarrow \mathbb{R}\ ,$$

$$H_2(\underline{x},\ y,\ S'') = \big(1 - \rho(S'')\big)H_1(\underline{x}) + \rho(S'')C\ ,$$

with $\rho : [A,\ +\infty[\longrightarrow [0, 1]$, $\rho \equiv 0$ on $[A,\ 2A]$, $\rho \equiv 1$ for $S'' \ge B - \epsilon$, ρ strictly increasing on $[2A,\ B-\epsilon]$, $\rho' \equiv \text{ct.} \in [\frac{1}{B-2A-\epsilon}, \frac{1}{B-2A-3\epsilon}]$ on $[2A+\epsilon,\ B-2\epsilon]$ (Figure 1 (2)). The symplectic form on $\widehat{M} \times \partial N \times [A,\ \infty[$ is $\omega' \oplus d(S''\lambda'')$, with λ'' the contact form on ∂N. We get

$$X_{H_2}(\underline{x},\ y,\ S'') = \big(1 - \rho(S'')\big)X_{H_1}(\underline{x}) - \big(C - H_1(\underline{x})\big)\rho'(S'')X''_{\mathrm{Reeb}}(y)\ .$$

The projection of a periodic orbit of X_{H_2} on $\widehat{M}$ is a periodic orbit of X_{H_1}. In particular, H_1 is constant along the projection. Moreover, the orbits appear on levels $S'' = \text{ct.}$ as there is no component $\frac{\partial}{\partial S''}$ in X_{H_2}. As a consequence, the coefficients in front of X_{H_1} and X''_{Reeb} are constant along one orbit of X_{H_2}. A 1-periodic orbit θ of X_{H_2} corresponds therefore to a couple $(\Gamma,\ \gamma)$ such that

- Γ is an orbit of X_{H_1} having period $1 - \rho(S'')$;

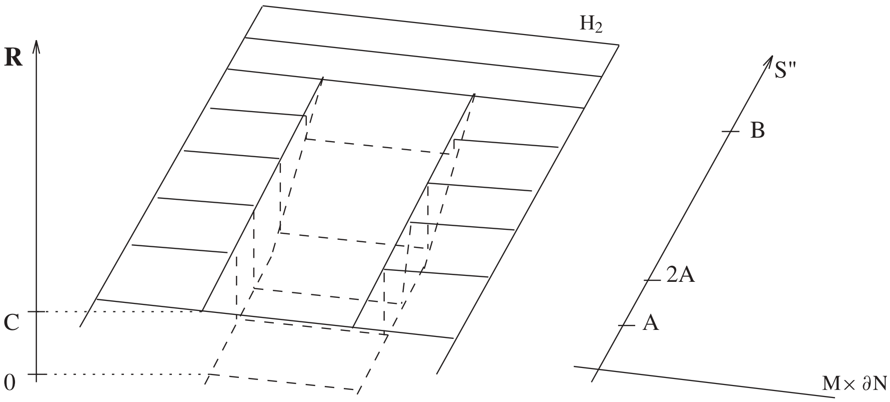


Fig. 2. Graph of the deformation H_2

- γ is a closed characteristic on the level $\partial N \times \{S''\}$, having period $\big(C - H_1(\underline{x})\big)\rho'(S'')$ and having the *opposite* orientation than the one given by X''_{Reeb}. We have used the notation $\underline{x} = \Gamma(0)$.

The action of θ is

$$\begin{aligned} A_{H_2+K_1}(\theta) &= -A(\Gamma) - A(\gamma) - H_2 - K_1 \\ &= A_{H_1}(\Gamma) - A(\gamma) - \rho(S'')\big(C - H_1(\underline{x})\big) - C \,. \end{aligned} \tag{9}$$

We have denoted by $A(\gamma)$, $A(\Gamma)$ the areas of the orbits γ and Γ respectively. The Hamiltonian K_1 is constant in the relevant domain and we have directly replaced it by its value C. It is useful to notice that $A(\gamma) = -S''\rho'(S'')(C - H_1(\underline{x}))$. The minus sign comes from the fact that the running orientation on γ is opposite to the one given by X''_{Reeb}. We have used the symplectic form $d(S''\lambda'')$ on the second factor in (9).

For $0 \leq T = 1 - \rho(S'') \leq 1$, the T-periodic orbits of X_{H_1} belong to one of the following classes.

1. constants in the interior of M, having zero area;
2. closed characteristics located around the level $S' = 1$, whose areas belong to the interval $[T_0(\partial M),\ T\lambda]$;
3. if $T\lambda \in \mathcal{S}(\partial M)$, one has a closed characteristic of area $S'T\lambda$ for any $S' \in [1 + \epsilon(\lambda),\ A - \epsilon(\lambda)]$ - the interval where H_1 is linear of s'lope λ;
4. closed characteristics located around the level $S' = A$, whose areas belong to the interval $[T_0(\partial M),\ T(\lambda - \eta_\lambda)A]$;
5. constants on levels $S' \geq A$, having zero area.

For $\epsilon > 0$ fixed we choose the various parameters involved in our constructions such that

$$\begin{aligned} \lambda(A-1) &\geq C \geq \lambda(A-1)-\epsilon, \\ \rho' &\leq 1/A(\sqrt{\lambda}-1)+\epsilon, \\ S'' \text{ close to } B &\implies \rho'(S'')\cdot S'' \leq \sqrt{\lambda}/(\sqrt{\lambda}-1)\,. \end{aligned}$$

We now show that the actions of 1-periodic orbits of X_{H_2} appearing in the region $\widehat{M}\times\partial N\times[A,\ \infty[$ tend uniformly to $-\infty$ when $\lambda\to+\infty$. We estimate the action of the orbits θ according to the type of their first component Γ and according to the level $S''\geq A$ on which lies γ.

1. Γ of type 1 corresponds to $H_1\in[-\epsilon,0]$.
   a) $S''\in[A,2A]\bigcup[B-\epsilon(\lambda),\ \infty[$. Because $\rho'=0$ there is no component of X_{H_2} in the X''_{Reeb} direction, orbits Γ appear in degenerate families (of dimension dim N) and the action of θ is $A_{H_2+K}(\theta)\leq\epsilon-C$.
   b) $S''\in[2A,\ \frac{A+B}{2}]$. Orbits Γ come in pairs with closed characteristics γ of period $\rho'(C-H_1(\underline{x}))$ on $\partial N\times\{S''\}$. We have

$$\begin{aligned} A_{H_2+K}(\theta) &\leq \epsilon+S''\rho'(S'')(C+\epsilon)-C \\ &\leq \epsilon+\frac{A+B}{2}\cdot\frac{1}{B-2A-3\epsilon}\cdot(C+\epsilon)-C \leq -\frac{1}{4}C\,. \end{aligned}$$

   The second inequality is valid for sufficiently large λ, in view of $B=A\sqrt{\lambda}$ which implies $(B+A)/(B-2A-3\epsilon)\to 1$.
   c) $S''\in[\frac{A+B}{2},B-\epsilon(\lambda)]$ (hence $\rho\in[\frac{1}{2},1]$). For λ big enough we have

$$\begin{aligned} A_{H_2+K}(\theta) &\leq \epsilon+S''\rho'(S'')(C+\epsilon)-\rho(S'')C-C \\ &\leq \epsilon+\frac{B-\epsilon}{B-2A-3\epsilon}\cdot(C+\epsilon)-\frac{1}{2}\cdot C-C \leq -\frac{1}{4}C\,. \end{aligned}$$

2. Γ of type 2 corresponds to $H_1\in[-\epsilon,\epsilon]$ and $S'\in[1,\ 1+\epsilon(\lambda)]$. The area of Γ belongs to the interval $[T_0(\partial M),\ (1-\rho(S''))\lambda]$.
   a) $S''\in[A,\ 2A]\bigcup[B-\epsilon(\lambda),\ \infty[$. As in 1a) we have

$$A_{H_2+K}(\theta) \leq \epsilon+(1-\rho(S''))\lambda-C \leq \epsilon+\lambda-C\,.$$

   b) $S''\in[2A,\ \frac{A+B}{2}]$. Like in 1b) the total action of θ is

$$A_{H_2+K}(\theta)\leq\epsilon+(1-\rho(S''))\lambda+S''\rho'(S'')(C+\epsilon)-C\leq-\frac{1}{2}C\,.$$

   c) $S''\in[\frac{A+B}{2},\ B-\epsilon(\lambda)]$. Following 1c) one has

$$\begin{aligned} A_{H_2+K}(\theta) &\leq \epsilon+(1-\rho(S''))\lambda+S''\rho'(S'')(C+\epsilon)-\rho(S'')C-C \\ &\leq -\frac{1}{4}C. \end{aligned}$$

3. Γ of type 3 has an action $A_{H_1}(\Gamma) \leq S'T\lambda - \lambda(S'-1-\epsilon') \leq (1+\epsilon')\lambda$, where $\epsilon' = \epsilon(\lambda)$.
   a) $S'' \in [A,\ 2A] \bigcup [B-\epsilon(\lambda),\ \infty[\ : A_{H_2+K}(\theta) \leq 2\lambda - C$.
   b) $S'' \in [2A,\ \frac{A+B}{2}] : A_{H_2+K}(\theta) \leq 2\lambda - \frac{1}{4}C$.
   c) $S'' \in [\frac{A+B}{2},\ B-\epsilon(\lambda)]$. The technique used in 1c) in order to get the upper bound no longer applies, as $C - H_1(\underline{x})$ can be arbitrarily close to 0. Nevertheless ρ satisfies by definition the inequality $(S''-2A)\rho'(S'') \leq \rho(S'')+\epsilon$. We thus get

$$
\begin{aligned}
&A_{H_2+K}(\theta)\\
&\leq (1+\epsilon')\lambda + S''\rho'(S'')(C - H_1(\underline{x})) - \rho(S'')(C - H_1(\underline{x})) - C\\
&\leq (1+\epsilon')\lambda + \frac{2A}{B-2A-3\epsilon}(C+\epsilon) + \epsilon(C+\epsilon) - C \leq 2\lambda - \frac{1}{2}C\,.
\end{aligned}
$$

4. Γ of type 4 corresponds to $H_1 \in [C-\epsilon,\ C]$ and $A_{H_1}(\Gamma) \leq (1-\rho(S''))\lambda A - \lambda(A-1) \leq \lambda$. In all three cases a)-c) we get $A_{H_2+K}(\theta) \leq 2\lambda - \frac{1}{4}C$.
5. Γ of type 5 corresponds to $H_1 \equiv C$. Like in 1a) there is no component in the X''_{Reeb} direction for X_{H_2} and orbits Γ appear in (highly) degenerated families. The total action in all three cases a) - c) is $A_{H_2+K}(\theta) = -C - C = -2C$.

This finishes the proof of the fact that the action of the new orbits of H_2 tends uniformly to $-\infty$.

d. The symmetric construction can be carried out for K in the region $\partial M \times [A,\ \infty[\times \widehat{N}$. One gets in the end a Hamiltonian $H_2 + K_2$ which is constant equal to $2C$ on $\{S' \geq B\} \bigcup \{S'' \geq B\}$. We modify now $H_2 + K_2$ outside the compact set $\{S' \leq B\} \bigcap \{S'' \leq B\}$ in order to make it linear with respect to the Liouville vector field $Z = X \oplus Y$ on $\widehat{M} \times \widehat{N}$. Let us define the following domains in $\widehat{M} \times \widehat{N}$ (see Figure 3):

$$
\begin{aligned}
&\mathbf{I} = \partial M \times [1,\ +\infty[\ \times \partial N \times [1,\ +\infty[\ ,\\
&\mathbf{II} = M \times \partial N \times [1,\ +\infty[\ ,\quad \mathbf{III} = \partial M \times [1,\ +\infty[\ \times N\ .
\end{aligned}
$$

Let $\Sigma \subset \widehat{M} \times \widehat{N}$ be a hypersurface which is transversal to Z such that

$$S'\ _{|\Sigma\,\cap\,\mathbf{III}} \equiv \alpha > 1, \quad S'\ _{|\Sigma\,\cap\,\mathbf{I}} \in [1,\ \alpha]\ ,$$

$$S''\ _{|\Sigma\,\cap\,\mathbf{II}} \equiv \beta > 1, \quad S''\ _{|\Sigma\,\cap\,\mathbf{I}} \in [1,\ \beta]\ .$$

We parameterize $\widehat{M} \times \widehat{N} \ \setminus\ \mathrm{int}(\Sigma)$ by

$$\Psi\ :\ \Sigma \times [1,\ +\infty[\longrightarrow \widehat{M} \times \widehat{N} \ \setminus\ \mathrm{int}(\Sigma)\ ,$$

$$(z,\,S) \longmapsto \left(\varphi'_{\ln S}(z),\ \varphi''_{\ln S}(z)\right)\ ,$$

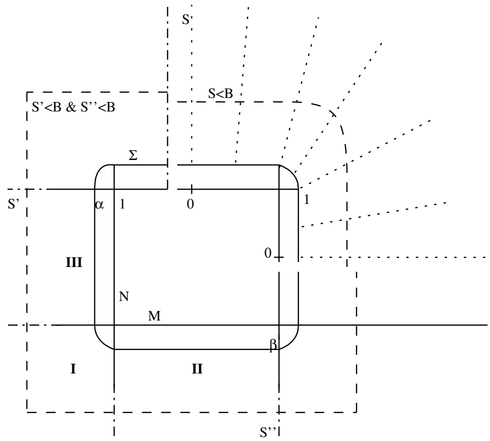


Fig. 3. Parameterization of the product $\widehat{M} \times \widehat{N}$

which is a symplectomorphism if one endows $\Sigma \times [1, +\infty[$ with the symplectic form $d(S\lambda|)$, where $\lambda| = \iota(X \oplus Y)\big(\omega \oplus \sigma\big)|_\Sigma$. As an example, for $z \in \Sigma \cap \mathbf{III}$ we have $\varphi'_{\ln S}(z) = \big(x(z),\ S\alpha\big)$. It is easy to see that

$$\Psi^{-1}\Big(\{S' \geq B\} \cup \{S'' \geq B\}\Big) \supseteq \{S \geq B\}\,. \tag{10}$$

As a consequence $H_2 + K_2$ is constant equal to $2C$ on $\{S \geq B\}$. We replace it by $L = l(S)$ on $\{S \geq B\}$, with l convex and $l'(S) = \mu \notin \mathcal{S}(\Sigma)$ for $S \geq B + \epsilon$. The additional 1-periodic orbits that are created in this way have action $A_L \leq \mu(B+\epsilon) - 2C = \mu(\sqrt{\lambda}A + \epsilon) - 2\lambda(A-1)$. By choosing $\mu = \sqrt{\lambda}$ one ensures $A_L \longrightarrow -\infty$, as well as the cofinality of the family of Hamiltonians L as $\lambda \to \infty$. Indeed, the Hamiltonian L is bigger than $(\sqrt{\lambda} - \epsilon)(S-1)$ on $\Sigma \times [1, \infty[$. Note that the choice of μ equal to $\sqrt{\lambda}$ and not belonging to the spectrum of Σ is indeed possible if we choose Σ to have a discrete and injective spectrum.

e. We have constructed a cofinal family of Hamiltonians $(L_\nu)_{\nu\geq 1}$ which are linear at infinity and that are associated to the initial Hamiltonians (H_ν) and (K_ν). I claim that the *Main Property* holds for $(L_\nu)_{\nu\geq 1}$. We assume of course that J^1_ν, J^2_ν and J_ν are regular almost complex structures for H_ν, K_ν, L_ν respectively, which are standard for $S' \geq 1+\epsilon$, $S'' \geq 1+\epsilon$ and $S \geq B+\epsilon$. Moreover, the almost complex structure J_ν is of the form $J_\nu = J^1_\nu \oplus J^2_\nu$ for $S \leq B$. The preceding estimates on the action of the 1-periodic orbits show that the sequence (8) of inclusions is certainly valid at the level of modules. This says in particular that the 1-periodic orbits involved in the free modules appearing in (8) are located in a neighbourhood of M, N and $M \times N$ respectively, and we can assume that the latter is contained in $\{S \leq 1\}$ for a suitable choice of Σ. Classical transversality arguments ([FH], Prop. 17 ; [FHS], 5.1 and 5.4) ensure that we can choose regular almost complex structures of the form described above. The point is now to prove that the inclusions (8) are also valid at the level of *differential* complexes.

It is enough to show that any trajectory $u = (v, w) : \mathbb{R} \times \mathbb{S}^1 \longrightarrow \widehat{M} \times \widehat{N}$ which satisfies $u(s, \cdot) \longrightarrow (x^{\pm}, y^{\pm})$, $s \longrightarrow \pm\infty$ stays in the domain $\{S \leq 1\}$, where the Floer equation is split. That will imply that v and w are Floer trajectories in $\widehat{M}$ and $\widehat{N}$ respectively and will prove (8) at the level of differential complexes. It is of course enough to prove this statement under the assumption that $J_\nu = J_\nu^1 \oplus J_\nu^2$ on the whole of $\widehat{M} \times \widehat{N}$: the Floer trajectories would then be a-posteriori contained in $\{S \leq 1\}$. Moreover, the proof of this fact only makes use of the split structure on the set $\{S' \leq 2A,\ S'' \leq 2A\}$ and this allows one to modify J_ν outside $\{S \leq B\}$ in order to formally work with an almost complex structure that is homothety-invariant at infinity.

We therefore suppose in the sequel that $J_\nu = J_\nu^1 \oplus J_\nu^2$. Arguing by contradiction, assume that the Floer trajectory u is not contained in $\{S \leq 1\}$. As u is anyway contained in a compact set, we infer that the function $S \circ u$ has a local maximum in $\{S > 1\}$, which means that one of the functions $S' \circ v$ or $S'' \circ w$ has a local maximum in $\{S' > 1 + \epsilon\}$ or $\{S'' > 1 + \epsilon\}$ respectively. The two cases are symmetric and we can assume without loss of generality that $S'' \circ w$ has a local maximum in $\{S'' > 1 + \epsilon\}$. In view of the fact that w satisfies the Floer equation associated to K_ν for $S'' \leq 2A$ the maximum principle ensures that the value of $S'' \circ w$ at the local maximum is in the interval $]2A, \infty[$. But $w(s, \cdot) \to y^{\pm}$, $s \to \pm\infty$ with $y^{\pm} \in \{S'' \leq 1 + \epsilon\}$ and this implies that w crosses the hypersurfaces $\{S'' = A\}$ and $\{S'' = 2A\}$. Moreover, the piece of w contained in $\{A \leq S'' \leq 2A\}$ is J_ν^2-holomorphic because K_ν is constant on that strip. We therefore obtain

$$\begin{aligned} A_{L_\nu}(x^-, y^-) - A_{L_\nu}(x^+, y^+) &= \int_{\mathbb{R}\times\mathbb{S}^1} |(v_s, w_s)|^2_{J_\nu^1 \oplus J_\nu^2} \\ = \int_{\mathbb{R}\times\mathbb{S}^1} |v_s|^2_{J_\nu^1} + |w_s|^2_{J_\nu^2} &\geq \int_{\mathbb{R}\times\mathbb{S}^1} |w_s|^2_{J_\nu^2} \\ \geq \int_{\left[(s,t)\ :\, w(s,t)\in\{A\leq S''\leq 2A\}\right]} |w_s|^2_{J_\nu^2} &= \mathrm{Area}(w \cap \{A \leq S'' \leq 2A\})\,. \end{aligned} \tag{11}$$

The last equality holds because w is J_ν^2-holomorphic in the relevant region. The lemma below will allow us to conclude. It is inspired from Hermann's work [He], where one can find it stated for $\widehat{N} = \mathbb{C}^n$.

Lemma 1. *Let (N, ω) be a manifold with contact type boundary and $\widehat{N}$ its symplectic completion. Let J be an almost complex structure which is homothety invariant on $\{S \geq 1\}$. There is a constant $C(J) > 0$ such that, for any $A \geq 1$ and any J-holomorphic curve u having its boundary components on both $\partial N \times \{A\}$ and $\partial N \times \{2A\}$, one has*

$$\mathrm{Area}(u) \geq C(J)A\,. \tag{12}$$

Proof. Consider, for $A \geq 1$, the map

$$\partial N \times [1,\, \infty[\xrightarrow{h_A} \partial N \times [1,\, \infty[\ ,$$

$$(p,\, S) \longmapsto (p,\, AS)\ .$$

By definition the map h_A is J-holomorphic. On the other hand, by using the explicit form of J given in §2, one sees that h_A expands the area element by a factor A. As a consequence, up to rescaling by h_A it is enough to prove (12) for $A = 1$. We apply Gromov's Monotonicity Lemma [G] 1.5.B, [Sik] 4.3.1 which ensures the existence of $\epsilon_0 > 0$ and of $c(\epsilon_0,\, J) > 0$ such that, for any $0 < \epsilon \leq \epsilon_0$, any $x \in \partial N \times [1,\, 2]$ with $B(x,\, \epsilon) \subset \partial N \times [1,\, 2]$ and any connected J-holomorphic curve S such that $x \in S$ and $\partial S \subset \partial B(x,\, \epsilon)$ one has

$$\mathrm{Area}(S \cap B(x,\, \epsilon)) \geq c(\epsilon_0,\, J)\epsilon^2\ .$$

Let us now fix ϵ small enough so that $B(x,\, \epsilon) \subset \partial N \times [1,\, 2]$ for all $x \in \partial N \times \{\frac{3}{2}\}$. As the boundary of u rests on both $\partial N \times \{1\}$ and $\partial N \times \{2\}$ one can find such a point x on the image of u. We infer $\mathrm{Area}(u) \geq \mathrm{Area}(u \cap B(x,\, \epsilon)) \geq c(\epsilon_0,\, J)\epsilon^2$. Then $C(J) = c(\epsilon_0,\, J)\epsilon^2$ is the desired constant. □

Applying Lemma 1 to our situation we get a constant $c > 0$ that does not depend on u and such that $\mathrm{Area}(w \cap \{A \leq S'' \leq 2A\}) \geq cA \geq C\lambda$. The difference of the actions in (11) is at the same time bounded by $4b$ and we get a contradiction for λ large enough.

For a fixed $b > 0$ the Floer trajectories corresponding to L_ν are therefore contained in $\{S \leq 1\}$ for ν large enough. This proves that the sequence of inclusions (8) is valid at the level of differential complexes. A few more commutative diagrams will now finish the proof. First, as a direct consequence of (8), one has the commutative diagram

$$\begin{tikzcd}
H_k\big(FC_*^{[-\delta,\, \frac{b}{2}[}(H_\nu,\, J_\nu^1) \otimes FC_*^{[-\delta,\, \frac{b}{2}[}(K_\nu,\, J_\nu^2)\big) \arrow[r] \arrow[d, dashed] & FH_k^{[-\delta,\, b[}(L_\nu,\, J_\nu) \arrow[d, dashed] \arrow[dl] \\
H_k\big(FC_*^{[-\delta,\, 2b[}(H_\nu,\, J_\nu^1) \otimes FC_*^{[-\delta,\, 2b[}(K_\nu,\, J_\nu^2)\big) \arrow[r] & FH_k^{[-\delta,\, 4b[}(L_\nu,\, J_\nu)
\end{tikzcd} \tag{13}$$

By taking the direct limit for $\nu \to \infty$ and $b \to \infty$ this induces the diagram

$$\begin{tikzcd}
\varinjlim\limits_{b \to +\infty} \varinjlim\limits_{\nu \longrightarrow +\infty} H_k\big(FC_*^{[-\delta,\, \frac{b}{2}[}(H_\nu,\, J_\nu^1) \otimes FC_*^{[-\delta,\, \frac{b}{2}[}(K_\nu,\, J_\nu^2)\big) \arrow[r] \arrow[d, dashed] & FH_k(M \times N) \arrow[d, dashed] \arrow[dl, "\sim"'] \\
\varinjlim\limits_{b \to +\infty} \varinjlim\limits_{\nu \longrightarrow +\infty} H_k\big(FC_*^{[-\delta,2b[}(H_\nu,\, J_\nu^1) \otimes FC_*^{[-\delta,2b[}(K_\nu,\, J_\nu^2)\big) \arrow[r] & FH_k(M \times N)
\end{tikzcd} \tag{14}$$

We easily see that *vertical arrows are isomorphisms* as the direct limits following ν and b commute with each other (this is a general property of bidirected systems). This implies that *the diagonal arrow is an isomorphism* as well. At the same time the algebraic Künneth theorem [D, VI.9.13] ensures the existence of a split short exact sequence

$$\begin{tikzcd} \bigoplus_{r+s=k} FH_r(M)\otimes FH_s(N) \arrow[r, tail] & \underset{b,\ \nu\to\infty}{\overrightarrow{\lim}} H_k\big(FC_*^{[-\delta,\,2b[}(H_\nu,\ J_\nu^1)\otimes FC_*^{[-\delta,\,2b[}(K_\nu,\ J_\nu^2)\big) \arrow[d, two heads] \\ & \bigoplus_{r+s=k-1} \mathrm{Tor}_1^A\big(FH_r(M),\ FH_s(N)\big) \end{tikzcd} \tag{15}$$

We infer the validity of the short exact sequence (1). We note that we use in a crucial way the exactness of the direct limit functor in order to obtain (15) from the Künneth exact sequence in truncated homology.

II. We prove now the existence of the morphism from the classical Künneth exact sequence to (1). We restrict the domain of the action to $[-\delta,\ \delta[$ with $\delta>0$ small enough. The Floer trajectories of a C^2-small autonomous Hamiltonian which is a Morse function on M coincide in the symplectically aspherical case with the gradient trajectories in the Thom-Smale-Witten complex. We denote by C_*^{Morse} the Morse complexes on the relevant manifolds. By (8) there is a commutative diagram

$$\begin{tikzcd} \bigoplus\limits_{r+s=k} FC_r^{[-\delta,2b[}(H_\nu,J_\nu^1)\otimes FC_s^{[-\delta,2b[}(K_\nu,J_\nu^2) \arrow[r, hook] & FC_k^{[-\delta,4b[}(L_\nu,J_\nu) \\ \bigoplus\limits_{r+s=k} C_{m+r}^{\mathrm{Morse}}(H_\nu,J_\nu^1)\otimes C_{n+s}^{\mathrm{Morse}}(K_\nu,J_\nu^2) \arrow[u] \arrow[r, equal] & C_{m+n+k}^{\mathrm{Morse}}(L_\nu,J_\nu^1\oplus J_\nu^2)\ . \arrow[u] \end{tikzcd} \tag{16}$$

The relevant Morse complexes compute homology relative to the boundary. With an obvious notation the above diagram induces in homology

$$\begin{tikzcd} \underset{b}{\overrightarrow{\lim}}\ \underset{n}{\overrightarrow{\lim}}\ H_k\big(FC_*^{[-\delta,\,2b[}(H_\nu,\ J_\nu^1)\otimes FC_*^{[-\delta,\,2b[}(K_\nu,\ J_\nu^2)\big) \arrow[r, "\sim"] & FH_k(M\times N) \\ H_{m+n+k}\big(C_{m+*}(M,\ \partial M)\otimes C_{n+*}(N,\partial N)\big) \arrow[u, "\phi"] \arrow[r, "\sim"] & H_{m+n+k}(M\times N,\ \partial(M\times N)) \arrow[u, "c_*"'] \end{tikzcd} \tag{17}$$

By naturality of the algebraic Künneth exact sequence, the map ϕ fits into the diagram below, where $*$ stands for its domain and target.

$$\begin{tikzcd}
\bigoplus_{r+s=k} FH_r(M) \otimes FH_s(N) \arrow[r, tail] & * \arrow[r, two heads] & \bigoplus_{r+s=k-1} \mathrm{Tor}_1^A\big(FH_r(M), FH_s(N)\big) \\
\bigoplus_{r+s=k} H_{m+r}(M, \partial M) \otimes H_{n+s}(N, \partial N) \arrow[r, tail] \arrow[u, "c_* \otimes c_*"] & * \arrow[r, two heads] \arrow[u, "\phi"] & \bigoplus_{r+s=k-1} \mathrm{Tor}_1^A\big(H_{m+r}(M, \partial M), H_{n+s}(N, \partial N)\big) \arrow[u, "\mathrm{Tor}_1(c_*)"]
\end{tikzcd} \tag{18}$$

Diagrams (17–18) establish the desired morphism of exact sequences. □

4. Applications

4.1. Computation of Floer homology groups

Proposition 2. *Let N be a compact symplectic manifold with restricted contact type boundary and let $\widehat{N}$ be its symplectic completion. Let $\widehat{N} \times \mathbb{C}^\ell$, $\ell \geq 1$ be endowed with the product symplectic form. Then*

$$FH_*(\widehat{N} \times \mathbb{C}^\ell) = 0 \, .$$

Proof. This follows directly from the Künneth exact sequence and from Floer, Hofer and Wysocki's computation $FH_*(\mathbb{C}^\ell) = 0$ [FHW]. □

We now fix terminology for the proof of Theorem C following [El]. A Stein manifold V is a triple (V, J_V, ϕ_V), where J_V is a complex structure and ϕ_V is an exhausting plurisubharmonic function. We say that V is of *finite type* if we can choose ϕ_V with all critical points lying in a compact set K. The Stein domains $V_c = \{\phi_V \leq c\}$ such that $V_c \supset K$ are called *big Stein domains* of ϕ_V ; they are all isotopic. We call (V, J_V, ϕ_V) *subcritical* if ϕ_V is Morse and all its critical points have indices strictly smaller than $\frac{1}{2} \dim_{\mathbb{R}} V$ (they are anyway at most equal to $\frac{1}{2} \dim_{\mathbb{R}} V$).

Let (V, J_V, ϕ_V) be a Stein manifold of finite type with ϕ_V Morse. Following [Se-Sm] we define a *finite type Stein deformation* of (V, J_V, ϕ_V) as a smooth family of complex structures J_t, $t \in [0, 1]$ together with exhausting plurisubharmonic functions ϕ_t such that: i) $J_0 = J_V$, $\phi_0 = \phi_V$; ii) the ϕ_t have only Morse or birth-death type critical points; iii) there exists c_0 such that all $c \geq c_0$ are regular values for ϕ_t, $t \in [0, 1]$. Condition ii) is not actually imposed in [Se-Sm], but we need it for the following theorem, which says that the existence of finite type Stein deformations on subcritical manifolds is a topological problem.

Theorem (compare [El, 3.4]). *Let (J_0, ϕ_0) and (J_1, ϕ_1) be finite type Stein structures on V. Assume J_0, J_1 are homotopic as almost complex structures and ϕ_0,*

ϕ_1 can be connected by a family ϕ_t, $t \in [0,1]$ of exhausting smooth functions which satisfy conditions ii)-iii) above and whose nondegenerate critical points have subcritical index for all $t \in [0,1]$. Then (J_0, ϕ_0), (J_1, ϕ_1) are homotopic by a finite type Stein deformation.

This is the finite type version of Theorem 3.4 in [El]. It holds because the latter is proved by h-cobordism methods within the plurisubharmonic category [El, Lemma 3.6], and these preserve the finite type condition.

Given a Stein manifold of finite type (V, J_V, ϕ_V), let us fix c_0 such that all $c \geq c_0$ are regular values of ϕ_V. Let $\omega_V = -d(d\phi_V \circ J_V)$. The big Stein domains $V_c = \{\phi_V \leq c\}$, $c \geq c_0$ are diffeomorphic and endowed with exact symplectic forms $\omega_c = \omega_V|_{V_c}$ for which ∂V_c is of restricted contact type.

The Floer homology groups $FH_*(V_c)$ are well defined and, by invariance under deformation of the symplectic forms, they are isomorphic [C1, 3.7]. We define $FH_*(V, J_V, \phi_V)$ as $FH_*(V_c)$ for c large enough.

Let now (J_t, ϕ_t), $t \in [0,1]$ be a finite type deformation on a Stein manifold V. Given c_0 such that all $c \geq c_0$ are regular values of ϕ_t for all $t \in [0,1]$, the big Stein domains $V_{t,c} = \{\phi_t \leq c\}$, $t \in [0,1]$, $c \geq c_0$ are all diffeomorphic and endowed with exact symplectic forms $\omega_{t,c} = -d(d\phi_t \circ J_t)$. The boundaries $\partial V_{t,c}$ are of restricted contact type and we can again apply Lemma 3.7 in [C1] to conclude that the Floer homology groups $FH_*(V_{t,c})$ are naturally isomorphic. In particular $FH_*(V, J_0, \phi_0) \simeq FH_*(V, J_1, \phi_1)$.

Proof of Theorem C. Cieliebak proved in [C2] that, given a subcritical Stein manifold of finite type $(\widehat{N}, J, \phi)$, there exists a Stein manifold of finite type (V, J_V, ϕ_V) and a diffeomorphism $F : V \times \mathbb{C} \longrightarrow \widehat{N}$ such that: i) $J_V \times i$ and F^*J are homotopic as almost complex structures; ii) $\phi_V + |z|^2$ and $F^*\phi$ are subcritical Morse functions with isotopic big Stein domains W_c, respectively W'_c. The functions $\phi_V + |z|^2$ and $F^*\phi$ are associated to handle decompositions $H_1 \cup \ldots \cup H_\ell$, $H'_1 \cup \ldots \cup H'_\ell$ of W_c and W'_c such that $\text{index}(H_s) = \text{index}(H'_s)$, $1 \leq s \leq \ell$ and the following property holds. Given $f \in \text{Diff}_0(V \times \mathbb{C})$ such that $f(W_c) = W'_c$, the attaching maps of $f \circ H_s$, H'_s, $2 \leq s \leq \ell$ are isotopic in $H'_1 \cup \ldots \cup H'_{s-1}$ (the condition is independent of f). Two such handle decompositions are called *isotopic*.

Because the handle decomposition determines the isotopy class of the associated Morse function and the two handle decompositions above are isotopic, we infer that the functions $\phi_V + |z|^2$ and $F^*\phi$ are isotopic, i.e. there exists $f \in \text{Diff}_0(V \times \mathbb{C})$ with $(F^*\phi) \circ f = \phi_V + |z|^2$.

We can therefore apply the previous theorem and conclude that the Stein structures $(J_V \times i, \phi_V + |z|^2)$ and $(F^*J, F^*\phi)$ can be connected by a finite type Stein deformation. It follows that $FH_*(V \times \mathbb{C}, F^*J, F^*\phi) \simeq FH_*(V \times \mathbb{C}, J_V \times i, \phi_V + |z|^2)$. By Proposition 2, the latter homology group vanishes. On the other hand, by invariance of Floer homology under symplectomorphism we have $FH_*(\widehat{N},\ J,\ \phi) \simeq FH_*(V \times \mathbb{C},\ F^*J,\ F^*\phi)$. □

4.2. Symplectic geometry in product manifolds

Proof of Theorem B. The statement follows readily from the existence of the commutative diagram given by the second part of Theorem A, taking into account the isomorphism $H^{2m}(M,\ \partial M) \otimes H^{2n}(N,\ \partial N) \xrightarrow{\sim} H^{2m+2n}(M \times N,\ \partial(M \times N))$, $2m = \dim M$, $2n = \dim N$. The latter is given by the usual Künneth formula in singular cohomology with coefficients in a field. □

Remark. Theorem B should be interpreted as a stability property for the SAWC condition.

Remark. Floer, Hofer and Viterbo [FHV] proved the Weinstein conjecture in a product $P \times \mathbb{C}^\ell$, $\ell \geq 1$ with P a closed symplectically aspherical manifold. The Weinstein conjecture for a product $\widehat{M} \times \widehat{N}$ with $\widehat{N}$ subcritical Stein and $\widehat{M}$ the completion of a restricted contact type manifold has been proved by Frauenfelder and Schlenk in [FrSc].

4.3. Symplectic capacities

The discussion below makes use of field coefficients. Let $\delta > 0$ be small enough. One defines (see e.g. [V]) the capacity of a compact symplectic manifold M with contact type boundary as

$$\begin{aligned} c(M) &= \inf\{b > 0\ :\ FH^m_{]-\delta,\, b]}(M) \longrightarrow H^{2m}(M,\ \partial M) \text{ is zero }\} \\ &= \sup\{b > 0\ :\ FH^m_{]-\delta,\, b]}(M) \longrightarrow H^{2m}(M,\ \partial M) \text{ is nonzero }\}\ . \end{aligned}$$

Here $2m = \dim M$. The next result is joint work with A.-L. Biolley, who applies it in her study of symplectic hyperbolicity [Bi].

Proposition 3. *Let M, N be compact symplectic manifolds with boundary of restricted contact type. Then*

$$c(M \times N) \leq 2 \min\big(c(M),\ c(N)\big)\ .$$

Proof. The Main Property (8) gives, for ν large enough and field coefficients, an arrow $\bigoplus_{r+s=m+n} FH^r_{]-\delta,\, \frac{b}{2}]}(H_\nu) \otimes FH^s_{]-\delta,\, \frac{b}{2}]}(K_\nu) \longleftarrow FH^{m+n}_{]-\delta, b]}(L_\nu)$.

Moreover, for fixed b and ν large enough the Hamiltonians H_ν, K_ν and L_ν compute the corresponding truncated cohomology groups of M, N and $M \times N$.

Like in Theorem A.b. we get the commutative diagram

$$\begin{tikzcd}
\bigoplus_{r+s=m+n} FH^r_{]-\delta,\,\frac{b}{2}]}(M) \otimes FH^s_{]-\delta,\,\frac{b}{2}]}(N) \arrow[d, "c_*^{b/2} \otimes c_*^{b/2}"] & FH^{m+n}_{]-\delta,\,b]}(M \times N) \arrow[l] \arrow[d, "c_*^b"] \\
H^{2m}(M,\,\partial M) \otimes H^{2n}(N,\,\partial N) & H^{2m+2n}(M \times N,\,\partial(M \times N)) \arrow[l, "\sim"']
\end{tikzcd}$$

Let now $b < c(M \times N)$. Then $c_*^b \neq 0$, hence $c_*^{b/2} \otimes c_*^{b/2} \neq 0$ and therefore $b/2 \leq \min\big(c(M),\, c(N)\big)$. □

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